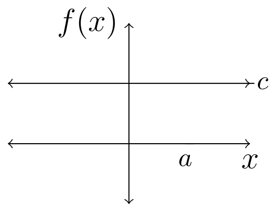
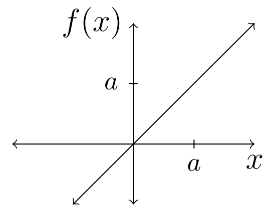
Objectives:

- Define some nice properties of limits (the "Limit Laws")
- Use limit laws to compute more complicated limits

Limits we know: (These are listed in your textbook as limit laws 7 and 8.)





$\lim_{x\to a} c = c$ e.g. $\lim_{x\to 3} 7 = 7$

$\lim_{x\to a} x = a$ e.g. $\lim_{x\to 4} x = 4$

Basic Limit Laws

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist: (This is a wildly important assumption!)

1. $\lim_{x\to a} \big(f(x) + g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a} \big(f(x) - g(x)\big) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. For any constant c, $\lim_{x\to a} \big(cf(x)\big) = c\left(\lim_{x\to a} f(x)\right)$
4. $\lim_{x\to a} \big(f(x)g(x)\big) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ ONLY IF $\lim_{x\to a} g(x)$ is not zero.

Some Limit Law Examples

1. $\lim_{x\to 2} (x+4) = \lim_{x\to 2} x + \lim_{x\to 2} 4 = 2 + 4 = 6$
2. $\lim_{h\to 100} (300 - h) = \lim_{h\to 100} 300 - \lim_{h\to 100} h = 300 - 100 = 200$
3. $\lim_{t\to 3} (7t) = 7\left(\lim_{t\to 3} t\right) = 7(3) = 21$
4. $\lim_{x\to 0} \left(x^2\right) = \lim_{x\to 0} (x)(x) = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0} x\right) = (0)(0)$
5. $\lim_{x\to 6} \dfrac{3}{x} = \dfrac{\lim_{x\to 6} 3}{\lim_{x\to 6} x} = \dfrac{3}{6} = \dfrac{1}{2}$ BUT we cannot use this limit law to compute $\lim_{x\to 0} \dfrac{3}{x}$ since $\lim_{x\to 0} x = 0$.

Building the Rest of the Limit Laws

6. For every positive integer n, $\lim_{x\to a}\left(f(x)\right)^n = \left(\lim_{x\to a} f(x)\right)^n$ because:

   $\lim_{x\to a}\left(f(x)\right)^n = \lim_{x\to a}\left(f(x)f(x)\dots f(x)\right) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} f(x)\right)\cdots\left(\lim_{x\to a} f(x)\right) = \left(\lim_{x\to a} f(x)\right)^n$

7. For every positive integer n, $\lim_{x\to a} x^n = \left(\lim_{x\to a} x\right)^n = a^n$

8. For every positive integer n, $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ (Note: if n is even, we need a to be positive)

9. For every positive integer n, $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ (if n is even, assume $\lim_{x\to a} f(x)$ is positive)

   because $\lim_{x\to a} f(x) = \lim_{x\to a}\left(\sqrt[n]{f(x)}\right)^n = \left(\lim_{x\to a}\sqrt[n]{f(x)}\right)^n$,

   so $\sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{\left(\lim_{x\to a}\sqrt[n]{f(x)}\right)^n} = \lim_{x\to a}\sqrt[n]{f(x)}$

Examples Involving Several Limit Laws

(a) Find $\lim_{x\to 1} 5x^{10} - 7x^3 - 8$.

$$\lim_{x\to 1}\left(5x^{10} - 7x^3 - 8\right) \overset{\textit{Limit Laws 1 \& 2}}{=} \lim_{x\to 1} 5x^{10} - \lim_{x\to 1} 7x^3 - \lim_{x\to 1} 8 \overset{\textit{Limit Law 3}}{=} 5\lim_{x\to 1} x^{10} - 7\lim_{x\to 1} x^3 - \lim_{x\to 1} 8$$

$$\overset{\textit{Limit Laws 6 \& 11}}{=} 5\left(\lim_{x\to 1} x\right)^{10} - 7x^3 - \lim_{x\to 1} 8 \overset{\textit{Limit Laws 7 \& 8}}{=} 5\left(1\right)^{10} - 7\left(1\right)^3 + 1 - 8 = 5 - 7 - 8 = -10$$

(b) Find $\lim_{x\to 3} \frac{2x^3 - 1}{x^2 + 6x}$.

$$\frac{\lim_{x\to 3}(2x^3 - 1)}{\lim_{x\to 3}(x^2 + 6x)} \overset{\textit{Limit Laws 1 \& 2}}{=} \frac{\lim_{x\to 3} 2x^3 - \lim_{x\to 3} 1}{\lim_{x\to 3} x^2 + \lim_{x\to 3} 6x} \overset{\textit{Limit Law 3}}{=} \frac{2\lim_{x\to 3} x^3 - \lim_{x\to 3} 1}{\lim_{x\to 3} x^2 + 6\lim_{x\to 3} x} \overset{\textit{Limit Law 6}}{=} \frac{2\left(\lim_{x\to 3} x\right)^3 - \lim_{x\to 3} 1}{\left(\lim_{x\to 3} x\right)^2 + 6\lim_{x\to 3} x} \overset{\textit{Limit Laws 7 \& 8}}{=} \frac{2\left(3\right)^3 - 1}{\left(3\right)^2 + 6(3)} =$$

$$\frac{2(27) - 1}{9 + 6(3)} = \frac{54 - 1}{9 + 18} = \frac{53}{27}$$

Conclusion: If $f(x)$ is a polynomial or rational function, and $f(a)$ is defined, then $\lim_{x\to a} f(x) =$ $f(a)$.

We say that these functions have the direct substitution property.

Example $\lim_{t\to 2} \frac{t^3 - t + 1}{t + 1} = \frac{2^3 - 2 + 1}{2 + 1} = \frac{8 - 1}{3} = \frac{7}{3}$

Example $\lim_{t\to -1} \frac{t^3 - t + 1}{t + 1}$ cannot be found using the direct substitution property, because -1 is not in the domain of $f(t) = \frac{t^3 - t + 1}{t + 1}$.